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A transformation formula for the generalized hypergeometric series ${}_4F_3(1)$

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Abstract
For positive a and b by employing representations for the Mellin transform of the product of two generalized hypergeometric functions ${}_1F_2(-a^2x^2)\,{}_1F_2(-b^2x^2)$, a new transformation formula for the series ${}_4F_3(1)$ is deduced.

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1. Introduction

In 1987, Wimp in his investigation of associated Jacobi polynomials deduced by means of Bailey's method for the case $p = 2$ [1, p 15] a transformation formula [7, lemma 2] for the generalized hypergeometric series ${}_{p+1}F_p((a_{p+1}); (b_p); 1)$. In 1997 Miller [3, equation (1.3b)] rederived Wimp's substantial result by means of rather elementary methods in lieu of contour integration and the calculus of residues, which were employed earlier.

In the present investigation we shall be concerned with the specialization $p = 3$ of the result alluded to above for ${}_{p+1}F_p(1)$. Thus we record

$$
\begin{aligned}
{}_4F_3\left(\begin{matrix} a, b, c, d \\ e, f, g \end{matrix}\right) &= \Gamma(e)\Gamma(f)\Gamma(g)\Gamma(1-d) \\
&\times \left\{ \frac{\Gamma(b-a)\Gamma(c-a)}{\Gamma(b)\Gamma(c)\Gamma(e-a)\Gamma(f-a)\Gamma(g-a)\Gamma(1+a-d)} \right. \\
&\times {}_4F_3\left(\begin{matrix} a, 1+a-e, 1+a-f, 1+a-g \\ 1+a-b, 1+a-c, 1+a-d \end{matrix}\right) \\
&+ \frac{\Gamma(a-b)\Gamma(c-b)}{\Gamma(a)\Gamma(c)\Gamma(e-b)\Gamma(f-b)\Gamma(g-b)\Gamma(1+b-d)} \\
&\times {}_4F_3\left(\begin{matrix} b, 1+b-e, 1+b-f, 1+b-g \\ 1+b-a, 1+b-c, 1+b-d \end{matrix}\right)
\end{aligned}
$$

$$+\frac{\Gamma(a-c)\Gamma(b-c)}{\Gamma(a)\Gamma(b)\Gamma(e-c)\Gamma(f-c)\Gamma(g-c)\Gamma(1+c-d)}$$
$$\times\, {}_4F_3\left(\begin{matrix} c, 1+c-e, 1+c-f, 1+c-g \\ 1+c-a, 1+c-b, 1+c-d \end{matrix}\right)\Bigg\} \tag{1.1}$$

where here and below we have suppressed the unit argument in each ${}_4F_3(1)$. However, we shall show that another transformation formula exists, namely

$$\begin{aligned} {}_4F_3\left(\begin{matrix} a, b, c, d \\ e, f, g \end{matrix}\right) &= \Gamma(e)\Gamma(f)\frac{\Gamma(1-c)\Gamma(1-d)}{\Gamma(1-g)} \\ &\times\left\{\frac{\Gamma(b-a)\Gamma(1+a-g)}{\Gamma(b)\Gamma(e-a)\Gamma(f-a)\Gamma(1+a-c)\Gamma(1+a-d)}\right. \\ &\times\, {}_4F_3\left(\begin{matrix} a, 1+a-e, 1+a-f, 1+a-g \\ 1+a-b, 1+a-c, 1+a-d \end{matrix}\right) \\ &+\frac{\Gamma(a-b)\Gamma(1+b-g)}{\Gamma(a)\Gamma(e-b)\Gamma(f-b)\Gamma(1+b-c)\Gamma(1+b-d)} \\ &\times\, {}_4F_3\left(\begin{matrix} b, 1+b-e, 1+b-f, 1+b-g \\ 1+b-a, 1+b-c, 1+b-d \end{matrix}\right) \\ &-\frac{\Gamma(g-1)\Gamma(1+a-g)\Gamma(1+b-g)}{\Gamma(a)\Gamma(b)\Gamma(g-c)\Gamma(g-d)\Gamma(1+e-g)\Gamma(1+f-g)} \\ &\left.\times\, {}_4F_3\left(\begin{matrix} 1+a-g, 1+b-g, 1+c-g, 1+d-g \\ 2-g, 1+e-g, 1+f-g \end{matrix}\right)\right\}. \end{aligned} \tag{1.2}$$

Obvious specializations of the parameters in equations (1.1) and (1.2) yield two three-term relations for Clausen's series ${}_3F_2(1)$ typified by equations (1)–(6) in [1, section 3.7].

2. Convergence criteria for $F(s)$

In order to derive equation (1.2) we consider for positive a and b the Mellin transform

$$F(s) \equiv \int_0^\infty x^{s-1}\, {}_1F_2\left(\begin{matrix} \alpha \\ \beta, \gamma \end{matrix}; -a^2x^2\right) {}_1F_2\left(\begin{matrix} \xi \\ \mu, \nu \end{matrix}; -b^2x^2\right) \mathrm{d}x \tag{2.1}$$

where for convergence of the integral at its lower limit the conditional inequality $\mathrm{Re}(s) > 0$ must hold.

Convergence of the integral at its upper limit may be determined by employing an asymptotic result for ${}_1F_2(-z^2)$ given in [4, equation (4.2)]. Thus we have for $|x| \to \infty$, $|\arg(x)| < \pi/2$

$$\begin{aligned} {}_1F_2\left(\begin{matrix} \alpha \\ \beta, \gamma \end{matrix}; -a^2x^2\right) {}_1F_2\left(\begin{matrix} \xi \\ \mu, \nu \end{matrix}; -b^2x^2\right) &= \Big\{ Ax^{-2\alpha-2\xi} + Bx^{\alpha+\xi-\beta-\gamma-\mu-\nu+1} \\ &\times \cos\left[2ax + \frac{\pi}{2}\left(\alpha-\beta-\gamma+\frac{1}{2}\right) + \mathcal{O}\left(\frac{1}{x}\right)\right] \\ &\times \cos\left[2bx + \frac{\pi}{2}\left(\xi-\mu-\nu+\frac{1}{2}\right) + \mathcal{O}\left(\frac{1}{x}\right)\right] \end{aligned}$$

$$+ Cx^{\alpha-\beta-\gamma-2\xi+\frac{1}{2}} \cos\left[2ax + \frac{\pi}{2}\left(\alpha - \beta - \gamma + \frac{1}{2}\right) + \mathcal{O}\left(\frac{1}{x}\right)\right]$$
$$+ Dx^{\xi-\mu-\nu-2\alpha+\frac{1}{2}} \cos\left[2bx + \frac{\pi}{2}\left(\xi - \mu - \nu + \frac{1}{2}\right) + \mathcal{O}\left(\frac{1}{x}\right)\right]\Bigg\}$$
$$\times \left[1 + \mathcal{O}\left(\frac{1}{x^2}\right)\right] \tag{2.2}$$

where the nonzero A, B, C and D are not only dependent on the parameters of each ${}_1F_2$, but are also proportional to powers of a and b. Since they are not important here, their forms are not displayed.

Noting that the product of two cosines in equation (2.2) is essentially proportional to a sum of two cosines, we need only determine for $z > 0$ the convergence of each integral on the right-hand side of

$$\int_z^\infty x^{s-1}{}_1F_2(-a^2x^2){}_1F_2(-b^2x^2)\,\mathrm{d}x$$
$$= A\int_z^\infty x^{s-2\alpha-2\xi-1}\,\mathrm{d}x + \frac{B}{2}\int_z^\infty x^{s+\alpha+\xi-\beta-\gamma-\mu-\nu}$$
$$\times \cos\left[2(a-b)x + \frac{\pi}{2}\,(\alpha - \xi + \mu + \nu - \beta - \gamma) + \mathcal{O}\left(\frac{1}{x}\right)\right]\mathrm{d}x$$
$$+ \frac{B}{2}\int_z^\infty x^{s+\alpha+\xi-\beta-\gamma-\mu-\nu}$$
$$\times \cos\left[2(a+b)x + \frac{\pi}{2}(\alpha + \xi - \mu - \nu - \beta - \gamma + 1) + \mathcal{O}\left(\frac{1}{x}\right)\right]\mathrm{d}x$$
$$+ C\int_z^\infty x^{s+\alpha-\beta-\gamma-2\xi-\frac{1}{2}} \cos\left[2ax + \frac{\pi}{2}\left(\alpha - \beta - \gamma + \frac{1}{2}\right) + \mathcal{O}\left(\frac{1}{x}\right)\right]\mathrm{d}x$$
$$+ D\int_z^\infty x^{s+\xi-\mu-\nu-2\alpha-\frac{1}{2}} \cos\left[2bx + \frac{\pi}{2}\left(\xi - \mu - \nu + \frac{1}{2}\right) + \mathcal{O}\left(\frac{1}{x}\right)\right]\mathrm{d}x \tag{2.3}$$

where for brevity we have suppressed the multiplicative factor $(1 + \mathcal{O}(1/x^2))$ in each integrand.

There are three cases to consider: $a \neq b$; $a = b$ which is called the *critical case* and $a = b$ where $\alpha - \xi + \mu + \nu - \beta - \gamma$ is an odd positive or negative integer N which is called the *supercritical case*. (See [5] and [6] for more on the critical and supercritical cases of the discontinuous integrals that are specializations of $F(s)$ defined by equation (2.1).)

It suffices to discuss only the convergence of the second integral I on the right-hand side of equation (2.3), since the other integrals are dealt with in the same way. (For details and references concerning justification for the convergence criteria below, see [3, section 3].) When $a \neq b$, I converges provided that $\mathrm{Re}(s + \alpha + \xi - \beta - \gamma - \mu - \nu) < 0$, but when $a = b$ it converges provided that $\mathrm{Re}(s + \alpha + \xi - \beta - \gamma - \mu - \nu) < -1$. In the supercritical case, however, for z sufficiently large the cosine in the integrand of I reduces to

$$\cos\left(\frac{\pi}{2}N\right)\cos\mathcal{O}\left(\frac{1}{x}\right) - \sin\left(\frac{\pi}{2}N\right)\sin\mathcal{O}\left(\frac{1}{x}\right) = \pm\mathcal{O}\left(\frac{1}{x}\right)$$

and so I converges provided that $\mathrm{Re}(s + \alpha + \xi - \beta - \gamma - \mu - \nu) < 0$.

Now taking into account the convergence of the other four integrals on the right-hand side of equation (2.3), we deduce the following.

Convergence criteria for $F(s)$: When $a \neq b$ or in the supercritical case $F(s)$ converges provided that

$$0 < \operatorname{Re} s < 2\operatorname{Re}(\alpha + \xi) \tag{2.4a}$$

$$0 < \operatorname{Re} s < \operatorname{Re}(\beta + \gamma + \mu + \nu - \alpha - \xi) \tag{2.4b}$$

$$0 < \operatorname{Re} s < \operatorname{Re}\left(\beta + \gamma + 2\xi - \alpha + \tfrac{1}{2}\right) \tag{2.4c}$$

$$0 < \operatorname{Re} s < \operatorname{Re}\left(\mu + \nu + 2\alpha - \xi + \tfrac{1}{2}\right). \tag{2.4d}$$

In the critical case when $a = b$ and $\alpha - \xi + \mu + \nu - \beta - \gamma$ is not an odd integer, $F(s)$ converges provided that

$$0 < \operatorname{Re} s < \operatorname{Re}(\beta + \gamma + \mu + \nu - \alpha - \xi - 1) \tag{2.4e}$$

and the inequalities (2.4*a*), (2.4*c*), (2.4*d*) hold true.

3. Representations for $F(s)$

Since the generalized hypergeometric function ${}_pF_q$ is a specialization of Meijer's G-function $G^{u,v}_{m,n}$ (see [2, p 129]), we may write

$${}_1F_2\left(\begin{matrix}\alpha\\ \beta, \gamma\end{matrix}; -a^2x^2\right) = \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha)} G^{1,1}_{1,3}\left(a^2x^2 \,\middle|\, \begin{matrix} 1-\alpha \\ 0, 1-\beta, 1-\gamma\end{matrix}\right).$$

Thus by using the well-known translation property of the G-function [2, p 69] and making the change in variables $x^2 = t$, we may write equation (2.1) as

$$F(s) = \frac{1}{2} a^{2-s} \frac{\Gamma(\beta)\Gamma(\gamma)\Gamma(\mu)\Gamma(\nu)}{\Gamma(\alpha)\Gamma(\xi)} \times \int_0^\infty G^{1,1}_{1,3}\left(a^2 t \,\middle|\, \begin{matrix} \frac{s}{2} - \alpha \\ \frac{s}{2} - 1, \frac{s}{2} - \beta, \frac{s}{2} - \gamma\end{matrix}\right) G^{1,1}_{1,3}\left(b^2 t \,\middle|\, \begin{matrix} 1-\xi \\ 0, 1-\mu, 1-\nu\end{matrix}\right) \mathrm{d}t.$$

The conditions for the convergence of the latter integral are also given by the inequalities (2.4).

The improper integral of a product of two arbitrary G-functions (when it exists) is itself proportional to a G-function (for this important result due to Meijer see [2, equation (3.10.11)]) and so we arrive at

$$F(s) = \frac{\Gamma(\beta)\Gamma(\gamma)\Gamma(\mu)\Gamma(\nu)}{\Gamma(\alpha)\Gamma(\xi)} a^{-s} G^{2,2}_{4,4}\left(\frac{b^2}{a^2} \,\middle|\, \begin{matrix} 1-\xi, 1-\frac{s}{2}, \beta - \frac{s}{2}, \gamma - \frac{s}{2} \\ 0, \alpha - \frac{s}{2}, 1-\mu, 1-\nu\end{matrix}\right) \tag{3.1a}$$

which can also be written as

$$F(s) = \frac{\Gamma(\beta)\Gamma(\gamma)\Gamma(\mu)\Gamma(\nu)}{\Gamma(\alpha)\Gamma(\xi)} b^{-s} G^{2,2}_{4,4}\left(\frac{a^2}{b^2} \,\middle|\, \begin{matrix} 1-\alpha, 1-\frac{s}{2}, \mu - \frac{s}{2}, \nu - \frac{s}{2} \\ 0, \xi - \frac{s}{2}, 1-\beta, 1-\gamma\end{matrix}\right). \tag{3.1b}$$

Equation (3.1*b*) can be obtained from (3.1*a*) and vice versa, since in equation (2.1) when a is interchanged with b, the respective parameters of each ${}_1F_2$ may be interchanged with each other, thus leaving $F(s)$ unchanged.

The function $G^{2,2}_{4,4}(z)$ where $|z| < 1$ may be written essentially as a sum of two generalized hypergeometric functions ${}_4F_3(z)$ (see [2, p 131]). Thus from equations (3.1*a*) and (3.1*b*) we deduce, respectively

$$F(s) = \frac{1}{2}a^{-s}\left\{\frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha)}\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\alpha-\frac{s}{2}\right)}{\Gamma\left(\beta-\frac{s}{2}\right)\Gamma\left(\gamma-\frac{s}{2}\right)}{}_4F_3\left(\begin{matrix}\xi, \frac{s}{2}, 1+\frac{s}{2}-\beta, 1+\frac{s}{2}-\gamma \\ \mu, \nu, 1+\frac{s}{2}-\alpha\end{matrix}; \frac{b^2}{a^2}\right)\right.$$
$$+\frac{\Gamma(\beta)\Gamma(\gamma)\Gamma(\mu)\Gamma(\nu)}{\Gamma(\xi)\Gamma(\beta-\alpha)\Gamma(\gamma-\alpha)}\frac{\Gamma\left(\frac{s}{2}-\alpha\right)\Gamma\left(\alpha+\xi-\frac{s}{2}\right)}{\Gamma\left(\alpha+\mu-\frac{s}{2}\right)\Gamma\left(\alpha+\nu-\frac{s}{2}\right)}\left(\frac{b^2}{a^2}\right)^{\alpha-\frac{s}{2}}$$
$$\left.\times {}_4F_3\left(\begin{matrix}\alpha, 1+\alpha-\beta, 1+\alpha-\gamma, \alpha+\xi-\frac{s}{2} \\ 1+\alpha-\frac{s}{2}, \alpha+\mu-\frac{s}{2}, \alpha+\nu-\frac{s}{2}\end{matrix}; \frac{b^2}{a^2}\right)\right\} \tag{3.2a}$$

where $0 < b < a$ and $\alpha - \frac{s}{2}$ may not be an integer or zero, and

$$F(s) = \frac{1}{2}b^{-s}\left\{\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\xi)}\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\xi-\frac{s}{2}\right)}{\Gamma\left(\mu-\frac{s}{2}\right)\Gamma\left(\nu-\frac{s}{2}\right)}{}_4F_3\left(\begin{matrix}\alpha, \frac{s}{2}, 1+\frac{s}{2}-\mu, 1+\frac{s}{2}-\nu \\ \beta, \gamma, 1+\frac{s}{2}-\xi\end{matrix}; \frac{a^2}{b^2}\right)\right.$$
$$+\frac{\Gamma(\beta)\Gamma(\gamma)\Gamma(\mu)\Gamma(\nu)}{\Gamma(\alpha)\Gamma(\mu-\xi)\Gamma(\nu-\xi)}\frac{\Gamma\left(\frac{s}{2}-\xi\right)\Gamma\left(\alpha+\xi-\frac{s}{2}\right)}{\Gamma\left(\beta+\xi-\frac{s}{2}\right)\Gamma\left(\gamma+\xi-\frac{s}{2}\right)}\left(\frac{a^2}{b^2}\right)^{\xi-\frac{s}{2}}$$
$$\left.\times {}_4F_3\left(\begin{matrix}\xi, 1+\xi-\mu, 1+\xi-\nu, \alpha+\xi-\frac{s}{2} \\ 1+\xi-\frac{s}{2}, \beta+\xi-\frac{s}{2}, \gamma+\xi-\frac{s}{2}\end{matrix}; \frac{a^2}{b^2}\right)\right\} \tag{3.2b}$$

where $0 < a < b$ and $\xi - \frac{s}{2}$ may not be an integer or zero.

Note that in equations (3.2) $\alpha + \xi - \frac{s}{2}$ can never be a negative integer or zero, since inequality (2.4*a*) must always hold true for the convergence of $F(s)$. Furthermore, the parameters β, γ, μ, ν may also never assume negative integer or zero values, since the generalized hypergeometric functions ${}_1F_2$ in equation (2.1) generally do not exist in this case.

Again by symmetry of variables a, b and concomitant parameters, the representations given for $F(s)$ by the right members of equations (3.2) may be obtained from each other. Nevertheless, the two representations are quite different and in fact are not analytic continuations of each other when $a \neq b$. Therefore the integral $F(s)$ is of a type called a discontinuous integral.

However, in the critical case not only do the inequalities (2.4*a*), (2.4*c*), (2.4*d*) and (2.4*e*) guarantee the convergence of $F(s)$, but also the inequality between the two right members of (2.4*e*) guarantees the convergence of each of the ${}_4F_3(1)$ in equations (3.2). Moreover, an argument very similar to that given in [3, section 3] for a related integral confirms that $F(s)$ is continuous across $a = b$ provided that the inequalities (2.4*a*), (2.4*c*), (2.4*d*) and (2.4*e*) hold true. In the critical case we are then permitted to equate the right members of equations (3.2).

We should mention that in the supercritical case when $a = b$, neither of the expressions on the right-hand side of equations (3.2) provide a valid representation for $F(s)$. This is so because it is required by (2.4*b*) that $0 < \operatorname{Re} s < \operatorname{Re}(\beta+\gamma+\mu+\nu-\alpha-\xi)$, but as we indicated above each of the ${}_4F_3(1)$ converges provided that $\operatorname{Re} s < \operatorname{Re}(\beta+\gamma+\mu+\nu-\alpha-\xi-1)$. Although the problem of finding a representation for $F(s)$ in the supercritical case is open, it is encouraging to know that a representation (that is valid in the critical and supercritical cases) in terms of Clausen's series ${}_3F_2(1)$ has recently been discovered for the specialization $F(s)|_{\alpha=\beta}$ (see [6, theorem 1]).

Although equations (3.1) provide representations for $F(s)$ in terms of Meijer's *G*-function, the *G*-function is essentially just an equivalent notation for a certain contour integral and so is not immediately useful computationally.

4. Transformation formula for ${}_4F_3(1)$

We have already noted in the previous section that in the critical case we are permitted to equate the right members of equations (3.2). Thus we obtain on replacing $s/2$ by s, suppressing the unit argument in each of the ${}_4F_3(1)$, and some rearrangement

$$\begin{aligned}
{}_4F_3\left(\begin{matrix}\xi, s, 1+s-\beta, 1+s-\gamma\\ \mu, \nu, 1+s-\alpha\end{matrix}\right) &= \Gamma(\mu)\Gamma(\nu)\frac{\Gamma(\beta-s)\Gamma(\gamma-s)}{\Gamma(\alpha-s)}\\
&\quad\times\left\{\frac{\Gamma(s-\xi)\Gamma(\alpha+\xi-s)}{\Gamma(s)\Gamma(\mu-\xi)\Gamma(\nu-\xi)\Gamma(\beta+\xi-s)\Gamma(\gamma+\xi-s)}\right.\\
&\quad\times{}_4F_3\left(\begin{matrix}\xi, 1+\xi-\mu, 1+\xi-\nu, \alpha+\xi-s\\ 1+\xi-s, \beta+\xi-s, \gamma+\xi-s\end{matrix}\right)\\
&\quad+\frac{\Gamma(\alpha)\Gamma(\xi-s)}{\Gamma(\beta)\Gamma(\gamma)\Gamma(\xi)\Gamma(\mu-s)\Gamma(\nu-s)}{}_4F_3\left(\begin{matrix}\alpha, s, 1+s-\mu, 1+s-\nu\\ \beta, \gamma, 1+s-\xi\end{matrix}\right)\\
&\quad-\frac{\Gamma(\alpha)\Gamma(s-\alpha)\Gamma(\alpha+\xi-s)}{\Gamma(s)\Gamma(\xi)\Gamma(\beta-\alpha)\Gamma(\gamma-\alpha)\Gamma(\alpha+\mu-s)\Gamma(\alpha+\nu-s)}\\
&\quad\left.\times{}_4F_3\left(\begin{matrix}\alpha, 1+\alpha-\beta, 1+\alpha-\gamma, \alpha+\xi-s\\ 1+\alpha-s, \alpha+\mu-s, \alpha+\nu-s\end{matrix}\right)\right\}.
\end{aligned}\tag{4.1}$$

Now setting $\xi = a, s = b, 1+b-\beta = c, 1+b-\gamma = d, \mu = e, \nu = f, 1+b-\alpha = g$, we deduce equation (1.2).

It is easily verified that each of the series ${}_4F_3(1)$ in equation (1.2) converges provided that $\mathrm{Re}(e+f+g-a-b-c-d) > 0$ which is equivalent under the latter transformations of parameters (including $s \mapsto 2s$ made earlier) to the right members of inequalities (2.4*e*), i.e., $\mathrm{Re}(\beta+\gamma+\mu+\nu-\alpha-\xi-s-1) > 0$. Moreover, the conditions given by the inequalities (2.4*a*), (2.4*c*) and (2.4*d*) have become superfluous with regard to equation (4.1) and therefore may be waved by appealing to the principle of analytic continuation.

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